Similitude laws in centrifuge modelling
Content

- Principle of scaling laws – Vashy-Bukingham theorem
- Scaling laws for centrifuge tests
- Scaling laws of water flow in centrifuge
- Grain size effects on interface and shear band pattern
Vaschy-Buckingham theorem

« If we have a physically meaningful equation involving a certain number, \( n \), of physical variables, and these variables are expressible in terms of \( k \) independent fundamental physical quantities, then the original expression is equivalent to an equation involving a set of \( p = n - k \) dimensionless parameters constructed from the original variables. »

\[
\begin{align*}
\mathbf{f}(X_1, X_2, \ldots, X_n) &= 0 \\
\text{for } i = 1, n \quad &X_i \rightarrow r_1^{\alpha_{1i}} \times r_2^{\alpha_{2i}} \times \cdots \times r_k^{\alpha_{ki}} \\
\text{for } i = 1, n-k \quad &\pi_i \rightarrow X_1^{\beta_{1i}} \times X_2^{\beta_{2i}} \times \cdots \times X_n^{\beta_{ni}} \\
\bar{\mathbf{f}}(\pi_1, \pi_2, \ldots, \pi_{n-k}) &= 0
\end{align*}
\]

Significance: Two systems for which these dimensionless parameters coincide are called similar (they differ only in scale); they are equivalent for the purposes of the equation.
• Application on the equation of dynamic equilibrium

\[
\text{div}\left(\sigma_p\right) + \rho_p \left(g_p + \frac{\partial^2}{\partial t^2} (\xi_p)\right) = 0
\]

- Direct method

| \(\sigma^* = \frac{\sigma_m}{\sigma_p}\) | \(\rho^* = \frac{\rho_m}{\rho_p}\) |
| \(g^* = \frac{g_m}{g_p}\) | \(t^* = \frac{t_m}{t_p}\) |
| \(\xi^* = \frac{\xi_m}{\xi_p}\) | \(L^* = \frac{L_m}{L_p}\) |

\[
\frac{\sigma^*}{\rho^* g^* L^*} = 1
\]

\[
\frac{\xi^*}{g^* t^{*2}} = 1
\]

\(n=6\)

\(k=3\)

- \(\sigma_p\): stress tensor
- \(u_p\): distance
- \(\rho_p\): density
- \(g_p\): volumic forces
- \(t_p\): time
- \(\xi_p\): displacement

L: meter
M: mass
T: time
- Second method

\[ f(\sigma, u, \rho, g, t, \xi) = 0 \]

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>T</th>
<th>L</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>1</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>( u )</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1</td>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>( g )</td>
<td>1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>( t )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \xi )</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Rank 3 \( \rightarrow p=3 \)

General expression of the non-dimensional parameters

\[ \pi_i = \sigma^{\alpha_i} \times u^{\alpha_{ui}} \times \rho^{\alpha_p} \times g^{\alpha_{gi}} \times t^{\alpha_{ti}} \times \xi^{\alpha_{\xi i}} \]

\[
\begin{align*}
\pi_1 &= 1 = \frac{\sigma^*}{\rho^* g^* h^*} \\
\pi_2 &= 1 = \frac{\xi^*}{g^* t^*} \\
\pi_3 &= 1 = \frac{\xi^*}{u^*}
\end{align*}
\]
Scalling law for reduce scale tests

Experimental work on geotechnical structures

- Models at reduce scale scaling factor $L^* = 1/n$
- Test at $1g: g^* = 1$
- Material with the same density: $\rho^* = 1$

$\xi^* = 1/n$
$\varepsilon^* = 1$
$\sigma^* = 1/n$
$t^* = 1/n$

Hooke law

$$\sigma_{ij} = \frac{E}{1+n\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

$$E' = \frac{1}{n}E$$

Mechanical characteristics of the material must be modified

$$1/n\sigma_{ij} = \frac{E}{1+n\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

$$\nu' = \nu$$
Some examples of 1g tests problem on reduced scale model

- Bearing capacity of shallow foundation (De Beer cited by Corté, 1989; Garnier, 2003)

Terzaghi (1943)

\[ q_u = \frac{1}{2} \gamma B N_\gamma (\phi) \]

Effect of B on Ny for cohesionless soils \( \Rightarrow K_n = \frac{q_u(B)/B}{q_u(B_0)/B_0} \)

Comabrieu (1997)

\[ q_u = \frac{1}{2} \gamma B N_\gamma (B) \]

\[ N_\gamma (B) = \frac{4A}{3\gamma} \left( \lambda + \frac{3}{2B} \right) \frac{\sin \phi (1 + \sin \psi)}{1 + \sin \phi} \]

Sol : \( E = E_0(1 + \lambda z) \)

(Garnier, 2003)
• Some example of 1g tests problem on reduced scale model

• Suction anchors (Puech A)

• Shallow foundation – reinforcement with a geotextil (Garnier 1995 – 1997)
• Centrifuge tests

- First idea: Phillips (1869) – France

- First tests: Bucky (1931) – USA
  Pokrovskii (1933) - URSS

1 scaling factor is fixed – same stress in the model and in the prototype

\[
\sigma^* = 1 + \text{Same soil} \\
\rho^* = 1
\]

\[
g^* = n
\]
• Scaling law for centrifuge tests

Technical committee 2 (TC2) of the ISSMGE


Available one line : http://www.tc2.civil.uwa.edu.au
Scalling law derived from the equation of equilibrium + $\sigma^* = 1$

| $\pi_1 = 1 = \frac{\sigma^*}{\rho^* g^* h^*}$ |
| $\pi_2 = 1 = \frac{\xi^*}{g^* t^*}$ |
| $\pi_3 = 1 = \frac{\xi^*}{u^*}$ |

<table>
<thead>
<tr>
<th>Scaling factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>distance</td>
</tr>
<tr>
<td>stress</td>
</tr>
<tr>
<td>soil density (same soil as prototype)</td>
</tr>
<tr>
<td>gravity</td>
</tr>
<tr>
<td>displacement</td>
</tr>
<tr>
<td>dynamic time</td>
</tr>
<tr>
<td>strain</td>
</tr>
<tr>
<td>velocity</td>
</tr>
<tr>
<td>acceleration</td>
</tr>
<tr>
<td>frequency</td>
</tr>
<tr>
<td>force</td>
</tr>
<tr>
<td>Unit weight</td>
</tr>
<tr>
<td>mass</td>
</tr>
</tbody>
</table>

$v^* = \frac{\xi^*}{t_{dyn}^*}$

$a^* = \frac{v^*}{t_{dyn}^*}$

$f^* = \frac{1}{t_{dyn}^*}$

$F^* = \sigma^* \times u^*^2$

$m^* = \rho^* \times u^*^3$

$\gamma^* = \rho^* \times g^*$
Vertical stress with depth in centrifuge tests

For a constant $\omega$, centrifuge acceleration varies with depth

$$\omega^2 = \frac{G}{R_n}$$

Minimum difference for $R_n = H/3$ below the soil surface

$$d\sigma_v = \rho G dr = \rho R \omega^2 dR$$

$$\sigma_v = \int_{R_0}^{R} \rho R \omega^2 dR = \frac{\rho G}{2R_n} \left( R^2 - R_0^2 \right)$$
Scaling law of water flow in centrifuge models

Navier Stokes equation (incompressible and Newtonien fluid)

\[
\frac{\partial u_p}{\partial t_p} + (u_p \cdot \text{grad}_p) \cdot u_p = g_p - \frac{1}{\rho_p} \text{grad}_p P_p + \nu_p \Delta_p u_p
\]

Dimensional analysis (direct method)

\[
\frac{\partial u_m}{\partial t_m} + (u_m \cdot \text{grad}_m) \cdot u_m = \frac{u_{fl}^2}{g \cdot x} g_m - \frac{\rho_{fl} u_{fl}^2}{P^*} \frac{1}{\rho_p} \text{grad}_p P_p + \frac{u_{fl}^* x^*}{\nu^*} \nu_p \Delta_p u_p
\]

Froude number

\[
F_r^* = \frac{u_{fl}^2}{g \cdot x} = 1
\]

Reynolds number

\[
R_e^* = \frac{u_{fl}^* x^*}{\nu^*} = 1
\]

\[
P^* = 1 \text{ and } \rho_{fl}^* = 1
\]

\[
u_p^* = 1
\]
• Scaling law of water flow in centrifuge models

Dimensional analysis (second method)

Hypothesis: not deformable porous media, incompressible and Newtonian fluid

\[ f(P, u_{fl}, d_p, L, \mu, \rho) = 0 \]

- \( P \): fluid pressure
- \( u_{fl} \): microscopic fluid velocity (interstitial velocity)
- \( d_p \): pore diameter
- \( L \): length
- \( \mu \): dynamic viscosity
- \( \rho \): density of the fluid

(Babendrier, 1991; Stephensen, 1979; Menand, 1995)

Friction factor

\[ F_f^* = \frac{i^* d_p^* g^*}{u_{fl}^*} \]

Reynolds number

\[ R_e^* = \frac{\rho^* u^* x^*}{\mu^*} = 1 \]
- Scaling law of water flow in centrifuge models

### Flow regimens
- Permanent flow
- Transient flow

### Flow medium
- Friction factor
  \[ F_f^* = \frac{i^* d^* g^*}{u_f^*} \]
- Reynolds number
  \[ Re^* = \frac{\rho^* u^* x^*}{\mu^*} \]

### Flow domain
- Forchheimer equation
  \[ u_f = K_f i \]
  \[ i = av_{fl} + bv_{fl}^2 + c \frac{\partial v_{fl}}{\partial t} \]

- Flow in porous media
- Creeping flow
  - Laminar flow without or with non-linear convective inertia forces
- Flow (ex: waves)
- Fully turbulent flow

- **Friction factor**
- **Reynolds number**
- **Forchheimer equation**
### Scaling law of water flow in centrifuge models

<table>
<thead>
<tr>
<th>Scaling factor</th>
<th>Scaling factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>distance</td>
<td>$L^*$</td>
</tr>
<tr>
<td>soil density</td>
<td>$\rho_{soil}$</td>
</tr>
<tr>
<td>gravity</td>
<td>$g^*$</td>
</tr>
<tr>
<td>dynamic time</td>
<td>$t_{dyn}^*$</td>
</tr>
<tr>
<td>velocity</td>
<td>$v^*$</td>
</tr>
</tbody>
</table>

- **Same fluid**
- **Same soil**

<table>
<thead>
<tr>
<th>Scaling factor</th>
<th>Scaling factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diffusion time</td>
<td>$T_{diff}^*$</td>
</tr>
<tr>
<td>Fluid velocity</td>
<td>$V_{fl}^*$</td>
</tr>
<tr>
<td>Dynamic viscosity</td>
<td>$\mu$</td>
</tr>
</tbody>
</table>

#### First approach

**Darcy’s formulation**

\[
v_{fl} = K_{fl}i = \frac{k g}{\mu} i
\]

\[
i = \frac{\Delta h}{L} \Rightarrow i^* = 1
\]

\[
K_{fl} = \frac{i}{v_{fl}} = n
\]

#### Second approach

**Hydraulic gradient express as a pressure**

\[
v_{fl} = K_{flp} \text{grad} P = \frac{k}{\mu} \text{grad} P
\]

\[
i^* = \frac{P^*}{L} = n
\]

\[
K_{flp} = \frac{i^*}{v_{fl}} = 1
\]
Scaling law of water flow in centrifuge models

Fluid flow in porous media - permanent flow regimen

<table>
<thead>
<tr>
<th>Same fluid</th>
<th>Same soil 2nd approach</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scaling factor</strong></td>
<td></td>
</tr>
<tr>
<td>distance</td>
<td>( L^* )</td>
</tr>
<tr>
<td>soil density</td>
<td>( \rho_{\text{soil}}^* )</td>
</tr>
<tr>
<td>gravity</td>
<td>( g^* )</td>
</tr>
<tr>
<td>dynamic time</td>
<td>( t^*_\text{dyn} )</td>
</tr>
<tr>
<td>velocity</td>
<td>( v^* )</td>
</tr>
</tbody>
</table>

Friction factor

\[
F_f^* = \frac{i^* d_i^* g^*}{v_{fl}^*} = \frac{n \times 1 \times n}{n} = n
\]

Reynolds number

\[
R_e^* = \frac{\rho^* v^* d_i^*}{\mu^*} = \frac{1 \times n \times 1}{1} = n
\]
Scaling law of water flow in centrifuge models

Static domain – permanent regimen: limit of validity of the Darcy law
(Khalifa et al., 2000; Goodings, 1994; Bezuijen, 2010; Wahuydy, 1998)

- Several definition of the reynolds number and friction factor in soils

\[
F_f = \frac{\text{ign}^3 d_{eq}}{3 \nu_n \tau^3 (1-n)} \quad R_e = \frac{2 \rho v_n \tau d_{eq}}{3 \mu (1-n)}
\]

(Khalifa et al., 2000)

\[
F_f = \frac{id_{50} gn^2}{v_f^2} \quad R_e = \frac{\rho v_f d_{50}}{n \mu}
\]

(Goodings, 1994)

\[
R_e = 3.9 - 5.5 \, (5\%)
\]

(Khalifa et al., 2000)

\[
R_e = 3.3
\]

(Goodings, 1994)
• Scaling law of water flow in centrifuge models

Static domain – permanent regimen: limit of validity of the Darcy law

Pokrovski & Fyodorov, 1975 : maximum hydraulic gradient for static geotechnical problems : \( i_{\text{max}} = 1 \)

<table>
<thead>
<tr>
<th></th>
<th>Fontainbleau</th>
<th>Labenne</th>
<th>Hostun</th>
<th>Le Rheu</th>
<th>Loire</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_{50} )</td>
<td>0.21</td>
<td>0.3</td>
<td>0.35</td>
<td>0.55</td>
<td>0.65</td>
</tr>
<tr>
<td>5% error</td>
<td>78</td>
<td>36</td>
<td>21</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>10% error</td>
<td>155</td>
<td>72</td>
<td>43</td>
<td>23</td>
<td>5</td>
</tr>
</tbody>
</table>

Maximum gravity level for similar behaviour between model and prototype (Khalifa et al., 2000)
Scaling law of water flow in centrifuge models

Dynamic domain \( \frac{\partial v}{\partial t} \neq 0 \) \rightarrow Non steady state flow

Same fluid
Same soil

<table>
<thead>
<tr>
<th>Dynamic time</th>
<th>Scaling factor</th>
<th>Diffusion time</th>
<th>Scaling factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>dynamic time</td>
<td>( t^{*}_{\text{dyn}} ) ( 1/n )</td>
<td>Diffusion time</td>
<td>( t^{*}_{\text{diff}} ) ( 1/n^2 )</td>
</tr>
<tr>
<td>velocity</td>
<td>( v^* ) ( 1 )</td>
<td>Fluid velocity</td>
<td>( v^{*}_{\text{fl}} ) ( n )</td>
</tr>
</tbody>
</table>

Forchheimer equation

\[
i = av_{\text{fl}} + bv_{\text{fl}}^2 + c \frac{\partial v_{fl}}{\partial t}
\]

2 stages

During the shaking: consolidation + dynamic strains
Dynamic loading of the soil mass: \( t^{*}_{\text{dyn}} = 1/n \)
Pore pressure dissipation: \( t^{*}_{\text{diff}} = 1/n^2 \)

After the shaking: consolidation
Pore pressure dissipation: \( t^{*}_{\text{diff}} = 1/n^2 \)
• Scaling law of water flow in centrifuge models

Stewart et al., 1998

Nondimensional time factor (consolidation)

\[ T = \frac{c_v t_{\text{diff}}}{h^2} = \frac{k m_v t_{\text{diff}}}{\mu h^2} \]

If \( \mu^* = n \), \( t_{\text{diff}}^* = t_{\text{dyn}}^* \)

Hypothesis: Darcy law valuable

30g centrifuge

\( \mu_{\text{HPMC}} = 10 \times \mu_{\text{water}} \)

\( v_{\mu} = \frac{k}{\mu} \text{grad}P \)

\( v_{\mu}^* = v^* = 1 \)

\( t_{\text{diff}}^* = t_{\text{dyn}}^* \)

\( \mu_m = n \times \mu_\rho \)

\( R_e^* = \frac{\rho^* v^* d_i^*}{\mu} = \frac{1 \times 1 \times 1}{n} = \frac{1}{n} \)

(Stewart et al., 1998)
• grain size effects on interfaces and shear band patterns

- Scale effects on shear mobilisation (Garnier & König, 1998)

Scaling effects?
- roughness effect : R/D50
- pile diameter effect : B/D50

(Garnier & König, 1998)
Roughness effect : R/D50

Normalized roughness: $R_n = \frac{R_{\text{max}}}{D_{50}}$

Shear box tests (steel/sand interface)

Centrifuge pull out test on vertical piles (b=12mm, D50=0.2mm 50g)

Three different zones of roughness

- **smooth interface**: interfacial shear along the soil-grain contact, small strength, no dilatancy
- Intermediate roughness: frictional resistance increases with the increase in roughness, low dilantency
- **Large roughness** (rough interface): internal shear localised in shear band into the sand friction angle independant from the roughness, large dilantency
Pile diameter effect: $B/D_{50}$

- Balachoski tests (1995)
  - Hostun sand $d_{50}=0.32$mm
  - Pile diameter $BB$: 16 to 55mm
  - Centrifuge acceleration: 100g
- LCPC tests
  - Fontainebleau sand $D_{50}=0.2$ mm
  - Pile diameter $B$ from 2 to 36 mm
  - Centrifuge acceleration: 50g

If $B/d_{50} > 100$ scale effect on $\tau_p$ is limited